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Valuation and dynamic replication of contingent claims in the framework of the beliefs-preferences gauge symmetry

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Abstract. Although symmetries play a major role in physics, their use in finance is relatively new and, to the best of our knowledge, can be traced to 1995 when Kholodnyi introduced the beliefs-preferences gauge symmetry. One of the main outcomes of the beliefs-preferences gauge symmetry is that it allows for the valuation and dynamic replication of contingent claims in a general market environment, that is, in the case of a general, not necessarily diffusion Markov process for the prices of underlying securities. This valuation and dynamic replication is based on the novel ideas of symmetry in contrast to the standard approach which uses stochastic analysis. The practical applications of the beliefs-preferences gauge symmetry range from the detection of a new type of true arbitrage to the beliefs-preferences-independent valuation and dynamic replication of contingent claims in a general market environment.

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1 Introduction

Although symmetries play a major role in physics, their use in finance is relatively new and, to the best of our knowledge, can be traced to 1995 when Kholodnyi introduced the beliefs-preferences gauge symmetry (see [4]).

The beliefs-preferences gauge symmetry establishes a fundamental symmetry between beliefs of market participants and their preferences in a general market environment, that is, in the case of a general, not necessarily diffusion Markov process for the prices of underlying securities.

One of the main outcomes of the beliefs-preferences gauge symmetry is that it allows for the beliefs-preferences-independent valuation and dynamic replication of contingent claims in a general market environment. This valuation and dynamic replication is based on the novel ideas of symmetry in contrast to the standard approach which uses stochastic analysis. Specifically, the beliefs-preferences gauge symmetry allows us to obtain an evolution equation that determines, in a general market environment, the values of European contingent claims that are independent of these beliefs and preferences. In the special case of the beliefs of market participants given by diffusion processes this evolution equation is reduced to the Black-Scholes equation with the dynamic replication given by the standard delta hedging. Being able to value and dynamically replicate European contingent claims one can value and dynamically replicate more general contingent claims such as universal contingent claims introduced by the author in [9–12].

In addition, with the help of the concept of a quasidifferential operator introduced by the author in [13], we propose the method of quasidifferential operators for the approximate valuation and dynamic replication of European contingent claims in a general market environment. We also observe that the method of quasidifferential operators can be viewed as a generalization of the method based on the Edgeworth expansion.

Based on the beliefs-preferences gauge symmetry, we show that randomness of the prices of underlying securities can be formalized as the noncommutativity of certain linear operators. In the case of the Black-Scholes market environment, these linear operators form a Lie algebra similar to that in quantum mechanics.

The practical applications of the beliefs-preferences gauge symmetry range from the detection of a new type of true arbitrage to the beliefs-preferences-independent valuation and dynamic replication of contingent claims in a general market environment.

2 Market environment

We present the framework of a market environment in which contingent claims are being priced that was introduced by the author in [11].

For the sake of financial clarity and in order not to obscure the structure of the beliefs-preferences gauge symmetry, which is essentially algebraic in nature, we present

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our results rather formally from the rigorous mathematical standpoint, supporting them whenever possible with appropriate financial justification. For the same reasons, we only consider the case of a single underlying security.

Consider an economy without transaction costs in which trading is allowed at any time in a *trading time set* $\mathcal{T}$, an arbitrary subset of the real numbers $\mathbb{R}$. Denote by $s_t > 0$ the unit price of the underlying security at time t in $\mathcal{T}$. Whenever ambiguity is unlikely, we will write s in place of s_t.

Denote by Π the vector space of all real-valued functions on the set of positive real numbers $\mathbb{R}_{++}$, and by Π_+ the nonnegative cone of Π consisting of all nonnegative real-valued functions on $\mathbb{R}_{++}$.

A *European option* with inception time t, expiration time T (t and T in $\mathcal{T}$ with $t \leq T$), and payoff g in Π_+ is a contract that gives the right but not the obligation to receive the payoff $g(s_T)$ at expiration time T, where the price of the underlying security is s_T at this time T.

A *European contingent claim* with inception time t, expiration time T (t and T in $\mathcal{T}$ with $t \leq T$), and payoff g in Π is a portfolio consisting of a long position on the European option with inception time t, expiration time T and payoff g^+, and a short position on the European option with inception time t, expiration time T and payoff g^-. Here g^+ and g^- in Π_+ are the nonnegative and nonpositive parts of g defined by $g^+(s) = \max\{g(s), 0\}$ and $g^-(s) = -\min\{g(s), 0\}$, so that $g = g^+ - g^-$.

For each t and T in $\mathcal{T}$ with $t \leq T$, denote by $\mathbf{V}(t,T)$ the operator that maps the payoff g of a European contingent claim with inception time t and expiration time T to its value $\mathfrak{E}(t,T,g) = \mathfrak{E}(t,T,g)(s_t)$ at inception time t as a function in Π of the price s_t of the underlying security at this time t:

$$\mathfrak{E}(t,T,g) = \mathbf{V}(t,T)g. \tag{1}$$

In this way $\mathbf{V}(t,T)$ performs both the valuation of a European contingent claim with a payoff g and the evolution in time of a payoff g, which justifies the following terminology.

For each t and T in $\mathcal{T}$ with $t \leq T$ we call [11] the operator $\mathbf{V}(t,T)$ on Π a *valuation* or *evolution operator*.

It is easy to see that, by the no-arbitrage argument, $\mathbf{V}(t,T)$ is a linear operator on Π that preserves the nonnegative cone Π_+ in Π, that is, $\mathbf{V}(t,T)$ is a nonnegative linear operator on Π. Moreover, $\mathbf{V}(t,T)$ is the identity operator on Π whenever $t = T$.

The evolution operators $\mathbf{V}(t,T)$, with t and T in $\mathcal{T}$ and $t \leq T$, contain all the information about the model of a market in which European contingent claims are being priced. This justifies the following terminology.

We will say [11] that a *market environment* is given by, or simply is the family $\mathbf{V} = \{\mathbf{V}(t,T) \,|\, t,T \in \mathcal{T}, t \leq T\}$ of evolution operators such that the following *intervention* or *intertemporal no-arbitrage* condition holds

$$\mathbf{V}(t,T) = \mathbf{V}(t,\tau)\mathbf{V}(\tau,T),$$

for each t, τ and T in $\mathcal{T}$ with $t \leq \tau \leq T$.

We call [11] a market environment $\mathbf{V}$ in which the evolution operators $\mathbf{V}(t,T)$ are functions of $T-t$ a *time-homogeneous market environment*.

The intervention condition financially expresses the requirement of intertemporal no-arbitrage in a derivative market and is a generalization to a general market environment of a semigroup intertemporal no-arbitrage condition introduced by Garman in [2] which, in our terminology, is applicable only to a time-homogeneous market environment.

A market environment $\mathbf{V}$ such that its trading time set $\mathcal{T}$ is an interval, either finite or infinite, of nonnegative real numbers admits the following characterization introduced by the author in [11].

We say that the one-parameter family $\boldsymbol{L} = \{\boldsymbol{L}(t) \,|\, t \in \mathcal{T}\}$ of linear operators on Π *generates* a market environment $\mathbf{V}$ if for each t and T in the trading time set $\mathcal{T}$ with $t \leq T$ and for each admissible payoff v_T in Π the function $\mathbf{V}(t,T)v_T$ of t is a solution, possibly generalized, of the Cauchy problem for the evolution equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}v + \boldsymbol{L}(t)v = 0, \quad t < T, \\ v(T) = v_T. \end{aligned} \tag{2}$$

An operator $\boldsymbol{L}(t)$ in the family $\boldsymbol{L}$ is called a *generator*.

If a market environment $\mathbf{V}$ is generated by the family of linear operators $\boldsymbol{L}$ then each evolution operator $\mathbf{V}(t,T)$ from $\mathbf{V}$ is formally given by the following product integral

$$\mathbf{V}(t,T) = \mathrm{e}^{\int_t^T \boldsymbol{L}(\tau)\mathrm{d}\tau}.$$

where each generator $\boldsymbol{L}(\tau)$ at time τ acts in the order opposite to that of τ. (For the definition of a product integral see, for example, [17]).

It is clear that according to the definition of the evolution operator $\mathbf{V}(t,T)$ in (1) the Cauchy problem in (2) determines the value of a European contingent claim with inception time t, expiration time T, and payoff v_T.

3 The Black-Scholes market environment

Following [11], we present an example of one of the major market environments, the Black-Scholes market environment, that corresponds to the Black-Scholes model.

We call a market environment $\mathbf{V}^{BS} = \{\mathbf{V}^{BS}(t,T)|t,T \in \mathcal{T}, t \leq T\}$ generated by the family $\boldsymbol{L}^{BS} = \{\boldsymbol{L}^{BS}(t)|t \in \mathcal{T}\}$ with the generators $\boldsymbol{L}^{BS}(t)$ defined by

$$\boldsymbol{L}^{BS}(t) = \frac{1}{2}\sigma^2(s,t)\,s^2\frac{\partial^2}{\partial s^2} + (r(s,t) - d(s,t))\,s\frac{\partial}{\partial s} - r(s,t)$$

a *Black-Scholes* market environment, where $\sigma(s,t)$ is the volatility, $r(s,t)$ is the continuously compounded interest rate, and $d(s,t)$ is the continuously compounded dividend yield in terms of the underlying security being a stock.

In the case of the Black-Scholes market environment, the evolution equation in (2) takes the form of the Black-Scholes equation

$$\frac{\partial}{\partial t}v + \frac{1}{2}\sigma^2(s,t)\,s^2\frac{\partial^2}{\partial s^2}v + (r(s,t) - d(s,t))\,s\frac{\partial}{\partial s}v - r(s,t)v = 0, \quad t < T, \qquad v(T) = v_T. \tag{3}$$

In the particular case when $\sigma(s,t)$, $r(s,t)$, and $d(s,t)$ are independent of the price s of the underlying security the evolution operators $\mathbf{V}^{BS}(t,T)$ are explicitely given by

$$(\mathbf{V}^{BS}(t,T)\,g)(s) = \frac{\mathrm{e}^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \times \int_0^\infty \mathrm{e}^{-(\log s'/s-(r-d-\sigma^2/2)(T-t))^2/2\sigma^2(T-t)}\,g(s')\frac{\mathrm{d}s'}{s'},$$

where g is an admissible payoff in Π, and where the effective volatility σ, effective interest rate r and effective dividend yield d are given by

$$\sigma^2 = \frac{1}{T-t}\int_t^T \sigma^2(\tau)\mathrm{d}\tau, \quad r = \frac{1}{T-t}\int_t^T r(\tau)\mathrm{d}\tau,$$

$$\text{and} \qquad d = \frac{1}{T-t}\int_t^T d(\tau)\mathrm{d}\tau.$$

4 Valuation of European contingent claims at equilibrium

Denote by $v(t) = v(t, s_t)$ in Π with t in $\mathcal{T}$ the value at inception time t of a European contingent claim with payoff $v(T) = v(T, s_T)$ in Π at expiration time T. At equilibrium (see, for example, [1]) the value $v(t) = v(t, s_t)$ is given by

$$v(t,s_t) = {u'}^{-1}(t,s_t)\int_0^\infty F(t,T,s_t,ds_T)\,u'(T,s_T)v(T,s_T), \tag{4}$$

where $u'(t, s_t) > 0$ denotes the marginal utility of consumption of a unit of account at time t when the price of the underlying security is s_t at this time t, and where $F(t, T, s_t, ds_T)$ stands for the transition probability of the prices of the underlying security.

Preceeding relationship (4) has the following financial interpretation. To carry one additional unit of a European contingent claim with the value $v(t, s_t)$ over the time period from t to T, $v(t, s_t)$ units of account have to be sacrificed resulting in the loss of utility $u'(t, s_t)v(t, s_t)$ at time t. By carrying this additional unit of the European contingent claim over the time period from t to T and selling it at time T, $v(T, s_T)$ additional units of account can be consumed resulting in the gain of utility $u'(T, s_T)v(T, s_T)$ at time T and, in turn, resulting in the expected gain of utility at time t

$$\int_0^\infty F(t,T,s_t,ds_T)\,u'(T,s_T)v(T,s_T).$$

At equilibrium the preceding expected gain of utility at time t has to be equal to the loss of utility $u'(t, s_t)v(t, s_t)$ at time t.

5 Formalization of the concept of a market participant

We call a *Markovian belief of a market participant*, or simply a *belief* the family $\boldsymbol{\mathcal{B}} = \{\boldsymbol{F}(t,T) \,|\, t, T \in \mathcal{T},\, t \le T\}$ of linear operators on Π defined for each admissible function h in Π by

$$\big(\boldsymbol{F}(t,T)\,h\big)(s_t) = \int_0^\infty F(t,T,s_t,ds_T)\,h(s_T),$$

such that

$$\boldsymbol{F}(t,\tau)\boldsymbol{F}(\tau,T) = \boldsymbol{F}(t,T),$$

for each t, τ, and T in $\mathcal{T}$ with $t \le \tau \le T$. By construction, $\boldsymbol{F}(t,T)$ is the identity operator on Π whenever $t = T$.

For the trading time set $\mathcal{T}$ being an interval, either finite or infinite, of nonnegative real numbers, a belief $\boldsymbol{\mathcal{B}}$ admits the following characterization.

We say that the family $\boldsymbol{\mathfrak{L}} = \{\boldsymbol{\mathfrak{L}}(t) \,|\, t \in \mathcal{T}\}$ of linear operators on Π *generates* a belief $\boldsymbol{\mathcal{B}}$ if for each t and T in $\mathcal{T}$ with $t \le T$ and for each admissible v_T in Π the function $\boldsymbol{F}(t,T)v_T$ of t is a solution, possibly generalized, of the following Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}v + \boldsymbol{\mathfrak{L}}(t)v = 0, \quad t < T, \qquad v(T) = v_T. \tag{5}$$

An operator $\boldsymbol{\mathfrak{L}}(t)$ in the family $\boldsymbol{\mathfrak{L}}$ is called a *generator*.

If a Markovian belief $\boldsymbol{\mathcal{B}}$ is generated by the family of linear operators $\boldsymbol{\mathfrak{L}}$ then each operator $\boldsymbol{F}(t,T)$ from $\mathcal{B}$ is formally given by the following product integral

$$\boldsymbol{F}(t,T) = \mathrm{e}^{\int_t^T \boldsymbol{\mathfrak{L}}(\tau)\mathrm{d}\tau},$$

where each generator $\boldsymbol{\mathfrak{L}}(\tau)$ at time τ acts in the order opposite to that of τ.

We call a *Markovian preference of a market participant*, or simply a *preference* the family $\boldsymbol{\mathcal{U}} = \{\boldsymbol{u'}(t) \,|\, t \in \mathcal{T}\}$ of linear operators on Π defined by

$$\big(\boldsymbol{u'}(t)\,h\big)(s_t) = u'(t,s_t)\,h(s_t),$$

for each admissible function h in Π.

Now we are ready to formalize the concept of a market participant in terms of their beliefs and preferences.

We call *a Markovian market participant*, or simply a *market participant* a pair $\boldsymbol{\mathcal{H}} = (\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{U}})$ where $\boldsymbol{\mathcal{B}}$ is a belief and $\boldsymbol{\mathcal{U}}$ is a preference.

We call *a Markovian market populace*, or simply a *market populace* the set $\mathbb{H} = \mathbb{B} \times \mathbb{U}$ of all market participants, where $\mathbb{B}$ is the set of *all beliefs of market participants* and $\mathbb{U}$ is the set of *all preferences of market participants.*

In the case when a market participant $\mathcal{H} = (\mathcal{B},\mathcal{U})$ has a belief $\mathcal{B}$ that is generated by $\mathfrak{L}$, then $\mathcal{H} = (\mathcal{B},\mathcal{U})$ can be uniquely characterized as a pair $(\mathfrak{L},\mathcal{U})$. In this case we will also write $\mathcal{H} = (\mathfrak{L},\mathcal{U})$. In the same way we will also write $\mathbb{H} = \mathbb{L} \times \mathbb{U}$, where $\mathbb{L}$ stands for the set of all families $\mathfrak{L}$.

6 Beliefs-preferences gauge symmetry group

We start by rewriting relationship (4) as follows

$$v(t) = \boldsymbol{u'}^{-1}(t)\boldsymbol{F}(t,T)\boldsymbol{u'}(T)\, v(T), \quad v(T) \in \Pi, \qquad (6)$$

where $\boldsymbol{F}(t,T)$ is in the belief $\mathcal{B}$ and $\boldsymbol{u'}(t)$ is in the preference $\mathcal{U}$ of a market participant $\mathcal{H} = (\mathcal{B},\mathcal{U})$.

Now we make the following crucial observation. Preceding relationship (6), by the no-arbitrage argument, has to hold for all market participants $\mathcal{H} = (\mathcal{B},\mathcal{U})$ in the market populace $\mathbb{H} = \mathbb{B} \times \mathbb{U}$ that populate the marketplace and for each admissible payoff $v(T)$ in Π. In this regard, for each pair of such market participants $\mathcal{H}_1 = (\mathcal{B}_1,\mathcal{U}_1)$ and $\mathcal{H}_2 = (\mathcal{B}_2,\mathcal{U}_2)$ in $\mathbb{H} = \mathbb{B} \times \mathbb{U}$ we have the following operator relationship

$$\boldsymbol{u}_1'^{-1}(t)\boldsymbol{F}_1(t,T)\boldsymbol{u}_1'(T) = \boldsymbol{u}_2'^{-1}(t)\boldsymbol{F}_2(t,T)\boldsymbol{u}_2'(T),$$

where $\boldsymbol{F}_1(t,T)$ is in $\mathcal{B}_1$ and $\boldsymbol{F}_2(t,T)$ is in $\mathcal{B}_2$, and where $\boldsymbol{u}_1'(t)$ is in $\mathcal{U}_1$ and $\boldsymbol{u}_2'(t)$ is in $\mathcal{U}_2$.

Therefore, the degree of freedom that market participants $\mathcal{H} = (\mathcal{B},\mathcal{U})$ in $\mathbb{H} = \mathbb{B} \times \mathbb{U}$ that populate the market place have in terms of their beliefs $\mathcal{B}$ and preferences $\mathcal{U}$ is determined by the following group of transformations $\mathcal{G}$ of the market populace $\mathbb{H} = \mathbb{B} \times \mathbb{U}$:

$$\mathcal{H} = (\mathcal{B},\mathcal{U}) \to g\,\mathcal{H} = \big(g_{\mathcal{B}}(\mathcal{B},\mathcal{U}), g_{\mathcal{U}}(\mathcal{B},\mathcal{U})\big), \quad g \in \mathcal{G}.$$

Here each $g = (g_{\mathcal{B}}, g_{\mathcal{U}})$ in $\mathcal{G}$ is defined by

$$\big(\boldsymbol{F}(t,T), \boldsymbol{u'}(t)\big) \xrightarrow{g_{\mathcal{B}}} \mathfrak{g}^{-1}(t)\boldsymbol{F}(t,T)\mathfrak{g}(T) \quad \text{and}$$
$$\big(\boldsymbol{F}(t,T), \boldsymbol{u'}(t)\big) \xrightarrow{g_{\mathcal{U}}} \mathfrak{g}^{-1}(t)\boldsymbol{u'}(t),$$

where $\boldsymbol{F}(t,T)$ is in $\mathcal{B}$ and $\mathfrak{g}^{-1}(t)\boldsymbol{F}(t,T)\mathfrak{g}(T)$ is in $g_{\mathcal{B}}(\mathcal{B},\mathcal{U})$, and where $\boldsymbol{u'}(t)$ is in $\mathcal{U}$ and $\mathfrak{g}^{-1}(t)\boldsymbol{u'}(t)$ is in $g_{\mathcal{U}}(\mathcal{B},\mathcal{U})$. Hereafter $\mathfrak{g}(t)$ is the linear operator on Π of multiplication by a positive function $\mathfrak{g}(t) = \mathfrak{g}(t, s_t)$ in Π. We comment that there is a certain restriction on the operators $\mathfrak{g}(t)$ on Π that we will discuss in the next section.

We call the group $\mathcal{G}$ the *beliefs-preferences gauge symmetry group for a market populace* $\mathbb{H}$. We will justify this terminology later in the article. (For the definition of a group, a gauge group and relevant terminology see, for example, [16]).

It is easy to see that the set of all market participants $\mathcal{H} = (\mathcal{B},\mathcal{U})$ in $\mathbb{H}$ that populate the marketplace is an orbit of the group $\mathcal{G}$ in $\mathbb{H}$. In this regard, we call such an orbit a *market generation.*

To the best of our knowledge, the existence of an interconnection between the beliefs and preferences of an investor was first noted by Samuelson and Merton (see [3]).

Finally, we conjecture that, in general, the absence of some type of arbitrage opportunity in a marketplace indicates the presence of a certain inherent symmetry, and, conversely, the presence of some type of arbitrage opportunity in a marketplace indicates the breaking of a certain inherent symmetry. Another example when this conjecture is true is the foreign exchange option symmetry introduced by Kholodnyi and Price in 1996 (see [5–8]).

7 The local in time representation of the beliefs-preferences gauge symmetry group

The beliefs-preferences gauge symmetry group $\mathcal{G}$ of transformations of the market populace $\mathbb{H} = \mathbb{B} \times \mathbb{U}$ can be represented as a group $\hat{\mathcal{G}}$ of transformations of $\mathbb{L} \times \mathbb{U}$:

$$(\mathfrak{L},\mathcal{U}) \to \hat{g}\,(\mathfrak{L},\mathcal{U}) = \big(\hat{g}_{\mathfrak{L}}(\mathfrak{L},\mathcal{U}), \hat{g}_{\mathcal{U}}(\mathfrak{L},\mathcal{U})\big), \quad \hat{g} \in \hat{\mathcal{G}}.$$

Here each $\hat{g} = (\hat{g}_{\mathfrak{L}}, \hat{g}_{\mathcal{U}})$ in $\hat{\mathcal{G}}$ is defined by

$$\mathfrak{L}(t) \xrightarrow{\hat{g}_{\mathfrak{L}}} \mathfrak{g}^{-1}(t)\,\mathfrak{L}(t)\,\mathfrak{g}(t) + \frac{\mathfrak{g}_t(t)}{\mathfrak{g}(t)} \quad \text{and}$$
$$\boldsymbol{u'}(t) \xrightarrow{\hat{g}_{\mathcal{U}}} \mathfrak{g}^{-1}(t)\boldsymbol{u'}(t),$$

where $\mathfrak{L}(t)$ is in $\mathfrak{L}$ and $\mathfrak{g}^{-1}(t)\,\mathfrak{L}(t)\,\mathfrak{g}(t) + \frac{\mathfrak{g}_t(t)}{\mathfrak{g}(t)}$ is in $\hat{g}_{\mathfrak{L}}(\mathfrak{L},\mathcal{U})$, and where $\boldsymbol{u'}(t)$ is in $\mathcal{U}$ and $\mathfrak{g}^{-1}(t)\boldsymbol{u'}(t)$ is in $\hat{g}_{\mathcal{U}}(\mathfrak{L},\mathcal{U})$. Hereafter a subscript variable denotes the corresponding partial derivative.

Because $\mathfrak{g}^{-1}(t)\,\mathfrak{L}(t)\,\mathfrak{g}(t) + \frac{\mathfrak{g}_t(t)}{\mathfrak{g}(t)}$ is the generator of a belief the function $\mathfrak{g}(t)$ has to be a positive solution of the evolution equation in (5).

Further in the article we will need the following relationship between the generators $\boldsymbol{L}(t)$ in $\boldsymbol{L}$ of a market environment $\mathbf{V}$ and the generators $\mathfrak{L}(t)$ in $\mathfrak{L}$ of the belief $\mathcal{B}$ as well as the preferences $\boldsymbol{u'}(t)$ in the preference $\mathcal{U}$ of a market participant $\mathcal{H} = (\mathcal{B},\mathcal{U})$:

$$\boldsymbol{L}(t) = \boldsymbol{u'}^{-1}(t)\,\mathfrak{L}(t)\,\boldsymbol{u'}(t) + \frac{\boldsymbol{u}_t'(t)}{\boldsymbol{u'}(t)}\,. \qquad (7)$$

The preceeding relationship directly follows from the following representation of the evolution operators $\mathbf{V}(t,T)$ in the market environment $\mathbf{V}$ in terms of the operators $\boldsymbol{F}(t,T)$ in the belief $\mathcal{B}$ as well as the preferences $\boldsymbol{u'}(t)$ in the preference $\mathcal{U}$ of a market participant $\mathcal{H} = (\mathcal{B},\mathcal{U})$:

$$\mathbf{V}(t,T) = \boldsymbol{u'}^{-1}(t)\boldsymbol{F}(t,T)\boldsymbol{u'}(T).$$

In turn, the preceeding relationship directly follows from (1) and (6).

8 Beliefs-preferences gauge symmetry for a market populace with beliefs determined by diffusion processes

Consider the beliefs-preferences gauge symmetry group $\hat{\mathcal{G}}$ for the market populace $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ in the particular case when each family $\mathfrak{L} = \{\mathfrak{L}(t) \,|\, t \in \mathcal{T}\}$ in $\mathbb{L}$ determines a diffusion process, that is, consists of generators $\mathfrak{L}(t)$ of the form

$$\mathfrak{L}(t) = \frac{1}{2}\sigma^2(s,t)\frac{\partial^2}{\partial s^2} + \mu(s,t)\frac{\partial}{\partial s}, \qquad (8)$$

where $\sigma^2(s,t)$ is the variance per unit time and $\mu(s,t)$ is the drift per unit time.

Our goal is to establish the behavior of the generator $\mathfrak{L}(t)$ under the transformations $\hat{g}$ from the group $\hat{\mathcal{G}}$, that is, to find the generator

$$\mathfrak{L}'(t) = \mathfrak{g}^{-1}(t)\,\mathfrak{L}(t)\,\mathfrak{g}(t) + \frac{\mathfrak{g}_t(t)}{\mathfrak{g}(t)}\cdot$$

It is easy to see that $\mathfrak{L}'(t)$ is also of the same form as $\mathfrak{L}(t)$:

$$\mathfrak{L}'(t) = \frac{1}{2}\sigma^2(s,t)\frac{\partial^2}{\partial s^2} + \mu'(s,t)\frac{\partial}{\partial s},$$

where

$$\mu'(s,t) = \mu(s,t) + \sigma^2(s,t)\frac{\mathfrak{g}_s(t,s)}{\mathfrak{g}(t,s)}\cdot \qquad (9)$$

Therefore, the behavior of the generator $\mathfrak{L}(t)$ under the transformations $\hat{g}$ from the group $\hat{\mathcal{G}}$ can be characterized solely in terms of the drift $\mu(s,t)$:

$$\mu(s,t) \to \mu(s,t) + \sigma^2(s,t)\frac{\mathfrak{g}_s(t,s)}{\mathfrak{g}(t,s)}\cdot$$

Finally, we note that the transformations $\hat{g}$ from the beliefs-preferences gauge symmetry group $\hat{\mathcal{G}}$ for the market populace $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ in the particular case when $\mathbb{L}$ is determined by diffusion processes are equivalent to the changes of measure due to the Girsanov theorem. (For the statement of the Girsanov theorem see, for example, [1]). In this regard, the transformations $\hat{g}$ from the beliefs-preferences gauge symmetry group $\hat{\mathcal{G}}$ for $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ with a general $\mathbb{L}$ can be viewed as a generalization of the changes of measure due to the Girsanov theorem to the case of a general Markov process.

9 Promoting the beliefs-preferences gauge symmetry

Our goal is to determine under what conditions European contingent claims in a general market environment can be valued *independently* of the beliefs $\mathfrak{L}$ in $\mathbb{L}$ and preferences $\boldsymbol{\mathcal{U}}$ in $\mathbb{U}$ of market participants $\boldsymbol{\mathcal{H}} = (\mathfrak{L}, \boldsymbol{\mathcal{U}})$ in the orbit of the group $\hat{\mathcal{G}}$ in $\mathbb{H} = \mathbb{L} \times \mathbb{U}$.

In order to have this independence of the values of European contingent claims we have to require that these values remain unchanged under the action of the group $G \times H$ of *independent* transformations of the beliefs $\mathfrak{L}$ in $\mathbb{L}$ and preferences $\boldsymbol{\mathcal{U}}$ in $\mathbb{U}$ of the market populace $\mathbb{H} = \mathbb{L} \times \mathbb{U}$:

$$\big(\mathfrak{L}(t), \boldsymbol{u}'(t)\big) \xrightarrow{\;g\;} \Big(\mathfrak{g}_1{}^{-1}(t)\,\mathfrak{L}(t)\,\mathfrak{g}_1(t) + \frac{\mathfrak{g}_{1t}(t)}{\mathfrak{g}_1(t)}, \boldsymbol{u}'(t)\Big),$$
$$\big(\mathfrak{L}(t), \boldsymbol{u}'(t)\big) \xrightarrow{\;h\;} \big(\mathfrak{L}(t), \mathfrak{g}_2{}^{-1}(t)\boldsymbol{u}'(t)\big),$$

where g and h are in G and H.

It is clear that the beliefs-preferences gauge symmetry group $\hat{\mathcal{G}}$ of transformations of $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ is the diagonal subgroup of the group $G \times H$ of transformations of $\mathbb{H} = \mathbb{L} \times \mathbb{U}$. In this regard, the desired independence of the values of European contingent claims of the beliefs $\mathfrak{L}$ in $\mathbb{L}$ and preferences $\boldsymbol{\mathcal{U}}$ in $\mathbb{U}$ of market participants $\boldsymbol{\mathcal{H}} = (\mathfrak{L}, \boldsymbol{\mathcal{U}})$ in the orbit of the group $\hat{\mathcal{G}}$ in $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ is equivalent to *promoting* the beliefs-preferences gauge symmetry group $\hat{\mathcal{G}}$ of transformations of $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ to the group $G \times H$ itself.

However, the values of European contingent claims do not, in general, remain unchanged under the action of the group $G \times H$. Indeed, it can be shown by substituting relationship (7) into the evolution equation in (2) that for a market participant $\boldsymbol{\mathcal{H}} = (\mathfrak{L}, \boldsymbol{\mathcal{U}})$ the value of a European contingent claim with inception time t, expiration time T, and payoff v_T is determined by the following Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}'(t)v + \mathfrak{L}(t)\boldsymbol{u}'(t)v = 0, \quad t < T,$$
$$v(T) = v_T, \qquad (10)$$

where $\mathfrak{L}(t)$ is in $\mathfrak{L}$ and $\boldsymbol{u}'(t)$ is in $\boldsymbol{\mathcal{U}}$.

Finally, we observe from the evolution equation in (10) that in order for the values of European contingent claims in a general market environment to remain unchanged under the action of the group $G \times H$ of transformations of $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ it is enough to require that these values remain unchanged under the actions of the groups G and H of transformations of $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ separately.

10 Dynamic replication of European contingent claims

In order to achieve the goal of finding the values of European contingent claims in a general market environment that remain unchanged under the action of the group $G \times H$, or equivalently, by the preceding discussion, H of transformations of the market populace $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ we choose [4] an alternative route.

Denote by Ω an index set which is assumed, for simplicity, to be a subset of nonnegative integers. For each t in $\mathcal{T}$ and ω in Ω let $\boldsymbol{\pi}(t,\omega)$ be the linear operator on Π of multiplication by a function $\pi(t,\omega) = \pi(t, s_t, \omega)$ in Π.

Let $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ be a set of traded European contingent claims with inception time t, expiration time T, payoffs $\{v_T^\omega : \omega \in \Omega\}$ and values $\{v_\omega = v_\omega(t) : \omega \in \Omega\}$ with the following property. For a European contingent claim $\mathfrak{v}$ with inception time t, expiration time T, payoff v_T and value $v = v(t)$ there exist a family of linear operators $\{\boldsymbol{\pi}(t,\omega) : \omega \in \Omega\}$ denoted by $\{\boldsymbol{\pi}_v(t,\omega) : \omega \in \Omega\}$ such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathfrak{L}(t)\right) \boldsymbol{u'}(t)\, v = \sum_{\omega\in\Omega} \boldsymbol{\pi}_v(t,\omega) \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathfrak{L}(t)\right) \boldsymbol{u'}(t)\, v_\omega, \quad t < T,$$
$$v(T) = v_T \tag{11}$$

for any preference $\boldsymbol{u'}(t)$. If in addition

$$v(t) = \sum_{\omega\in\Omega} \boldsymbol{\pi}_v(t,\omega)\, v_\omega(t), \quad t < T, \tag{12}$$

we will say that the set $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ of the European contingent claims *dynamically replicates* the European contingent claim $\mathfrak{v}$. We will call the portfolio of the European contingent claims $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ with *operator weights* $\{\boldsymbol{\pi}_v(t,\omega) : \omega \in \Omega\}$ a *dynamically replicating portfolio*. We comment that a portfolio of the European contingent claims $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ with operator weights $\{\boldsymbol{\pi}_v(t,\omega) : \omega \in \Omega\}$ is understood as a portfolio with weights $\{\pi_v(t, s_t, \omega) : \omega \in \Omega\}$ that depend on time t and the price s_t of the underlying security at this time t.

The financial justification for the preceeding terminology is as follows. It is clear that the values $\{v_\omega = v_\omega(t) : \omega \in \Omega\}$ and $v = v(t)$ of the European contingent claims $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ and $\mathfrak{v}$ determined by the Cauchy problem in (10) with the final conditions $\{v_\omega(T) = v_T^\omega : \omega \in \Omega\}$ and $v(T) = v_T$ do not separately remain unchanged under the action of the group H. However, if for a European contingent claim $\mathfrak{v}$ there exists a set $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ of European contingent claims that dynamically replicates $\mathfrak{v}$ then the value

$$v - \sum_{\omega\in\Omega} \boldsymbol{\pi}_v(t, d\omega) v_\omega$$

of the portfolio consisting of a long position on the European contingent claim $\mathfrak{v}$ and a short position on its dynamically replicating portfolio is such a solution of the Cauchy problem in (10) that it remains unchanged under the action of the group H. In order to see this it is enough to rewrite the Cauchy problem in (11) as follows

$$\left(\frac{\overset{2}{\mathrm{d}}}{\mathrm{d}t} + \mathfrak{L}(\overset{2}{t})\right) \left(\boldsymbol{u'}(\overset{1}{t}) \left(v - \sum_{\omega\in\Omega} \overset{3}{\boldsymbol{\pi}}_v(t,\omega) v_\omega\right)\right) = 0, \quad t < T,$$
$$v(T) = v_T$$

for any preference $\boldsymbol{u'}(t)$. Here the numbers over the operators indicate the order in which these operators act. (For an introduction to noncommutative analysis see, for example, [18].)

Finally, we call a market environment $\mathbf{V}$, or simply a market, *dynamically complete* if for each t and T in $\mathcal{T}$ with $t < T$ there exists a *dynamically spanning* set $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ of European contingent claims with inception time t, expiration time T and payoffs $\{\mathfrak{v}_T^\omega : \omega \in \Omega\}$ that dynamically replicates each European contingent claim with inception time t, expiration time T and the payoff v_T in some dense set of payoffs.

11 Beliefs-preferences-independent valuation of European contingent claims

Let the pure discount bond be in the set $\{\mathfrak{v}_\omega : \omega \in \Omega\}$, that is, let there exist ω_B in Ω such that the European contingent claim $\mathfrak{v}_{\omega_B}$ has inception time t, expiration time T, and payoff $\mathbf{1}$, the function in Π identically equal to unity.

It can be shown [4] that the Cauchy problem in (11) can be represented in such a way that the preference $\boldsymbol{u'}(t)$ does not enter it explicitly:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathfrak{L}(t)\right) v = \sum_{\omega\in\Omega\setminus\omega_B} \boldsymbol{\pi}_v(t,\omega) \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathfrak{L}(t)\right) v_\omega + \left(v - \sum_{\omega\in\Omega\setminus\omega_B} \boldsymbol{\pi}_v(t,\omega)\, v_\omega\right) r(s,t), \quad t < T,$$
$$v(T) = v_T, \tag{13}$$

where $r(s,t)$ is the interest rate. This is the sought after equation that determines the values of dynamically replicated European contingent claims in a general market environment. In this equation the operator weights $\{\boldsymbol{\pi}_v(t,\omega) : \omega \in \Omega\setminus\omega_B\}$ of the European contingent claims $\{\mathfrak{v}_\omega : \omega \in \Omega\setminus\omega_B\}$ in the dynamically replicating portfolio for the European contingent claim $\mathfrak{v}$ are determined by the following equation:

$$[[\mathfrak{L}(t), \boldsymbol{v}], \boldsymbol{u'}(t)]\, \mathbf{1} = \sum_{\omega\in\Omega\setminus\omega_B} \boldsymbol{\pi}_v(t,\omega) [[\mathfrak{L}(t), \boldsymbol{v}_\omega], \boldsymbol{u'}(t)]\, \mathbf{1}, \tag{14}$$

for any preference $\boldsymbol{u'}(t)$. Here $[A, B] = AB - BA$ is the commutator of linear operators A and B. The operator weight $\boldsymbol{\pi}_v(t, B) = \boldsymbol{\pi}_v(t, \omega_B)$ of the pure discount bond is given according to relation (12) by

$$\boldsymbol{\pi}_v(t, B) = \boldsymbol{v} - \sum_{\omega\in\Omega\setminus\omega_B} \boldsymbol{\pi}_v(t,\omega)\, \boldsymbol{v}_\omega. \tag{15}$$

We comment that in the derivation of the evolution equation in (13) we expressed the price of the pure discount bond in terms of the interest rate $r(s,t)$ and used noncommutative analysis.

Finally, we note that being able to value and dynamically replicate European contingent claims one can value and dynamically replicate more general contingent claims such as universal contingent claims introduced by the author in [9–12].

12 Noncommutativity as the formalization of randomness of the prices of the underlying security

Preceeding equation (14) and relationship (15) determine the map $\boldsymbol{\pi}_{(\cdot)}(t,\omega)$ defined on Π for each t in $\mathcal{T}$ and ω in Ω. More precisely, $\boldsymbol{\pi}_v(t,\omega)$ is a linear operator on Π of multiplication by the function $\pi_v(t,\omega)$ in Π for each admissible v in Π and for each t in $\mathcal{T}$ and ω in Ω. In this regard, define the operator $\hat{\pi}(t,\omega)$ on Π by

$$\hat{\pi}(t,\omega)\,v = \pi_v(t,\omega), \quad t \in \mathcal{T},\ \omega \in \Omega.$$

By the no-arbitrage argument, the operator $\hat{\pi}(t,\omega)$ is linear for each t in $\mathcal{T}$ and ω in Ω. We will refer [4] to each linear operator $\hat{\pi}(t,\omega)$ as a *portfolio operator*.

It can be shown (see [4] and [14]) that, in general, the portfolio operators $\hat{\pi}(t,\omega)$ on Π and linear operators on Π of multiplication by arbitrary admissible functions in Π do not commute. It turns out that it is this noncommutativity that in our formalism is responsible for the randomness of the prices of the underlying security.

More precisely, the portfolio operators $\hat{\pi}(t,\omega)$ commute with linear operators on Π of multiplication by arbitrary admissible functions in Π if and only if the generators $\mathfrak{L}(t)$ in $\mathfrak{L}$ of the belief $\boldsymbol{\mathcal{B}}$ of a market participant $\boldsymbol{\mathcal{H}} = (\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{U}})$ are of the form

$$\mathfrak{L}(t) = \mu(s,t)\frac{\partial}{\partial s},$$

where $\mu(s,t)$ is an admissible real-valued function. In turn, according to relationship (9), this is also true for all market participants in the same market generation as $\boldsymbol{\mathcal{H}} = (\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{U}})$.

Equivalently, the portfolio operators $\hat{\pi}(t,\omega)$ commute with linear operators on Π of multiplication by arbitrary admissible functions in Π if and only if the operator weights $\boldsymbol{\pi}_v(t,\omega)$ of the European contingent claims $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ in the dynamically replicating portfolio for each European contingent claim $\mathfrak{v}$ can be chosen in such a way that

$$\boldsymbol{\pi}_v(t,\omega) = \mathbf{0},$$

for each $\omega \neq \omega_B$. In other words, the portfolio operators $\hat{\pi}(t,\omega)$ commute with linear operators on Π of multiplication by arbitrary admissible functions in Π if and only if pure discount bonds dynamically span the market.

Finally, we comment that the noncommutativity of the portfolio operators $\hat{\pi}(t,\omega)$ with the linear operators on Π of multiplication by arbitrary admissible functions in Π has the following implication in financial modeling. The only parameters directly observable in a marketplace are those that depend on the actual price of the underlying security at a current time, that is, are functions of this price. Therefore, the only portfolio operators $\hat{\pi}(t,\omega)$ that are directly observable in a marketplace are the linear operators on Π of multiplication by arbitrary admissible functions in Π. In this regard, the only market environment that can be described solely in terms of the parameters directly observable in a marketplace is the one that is dynamically spanned by pure discount bonds, that is, in which the dynamics of the prices of the underlying security is non-random. If a market environment is not dynamically spanned by pure discount bonds, that is, if the dynamics of the prices of the underlying security is random then the market environment can not be described solely in terms of the parameters directly observable in a marketplace. In other words, if a market environment is not dynamically spanned by pure discount bonds, that is, if the dynamics of the prices of the underlying security is random then one needs a model for the dynamics of the prices of the underlying security.

13 The Black-Scholes equation as a special case

Consider the Cauchy problem in (13) in the particular case of the beliefs of market participants determined by diffusion processes, that is, with the generators $\mathfrak{L}(t)$ of the form (8).

For each European contingent claim $\mathfrak{v}$ with inception time t, expiration time T and payoff v_T we assume that the set $\{\mathfrak{v}_\omega : \omega \in \Omega\}$ of European contingent claims that dynamically replicates $\mathfrak{v}$ consists of the pure discount bond $\mathfrak{v}_{\omega_B}$ and the underlying security $\mathfrak{v}_{\omega_S}$ itself, that is, the set Ω consists of two elements ω_B and ω_S. We view the underlying security $\mathfrak{v}_{\omega_S}$ as the European contingent claim with inception time t, expiration time T, and payoff $v_T^{\omega_S}$ such that $v_T^{\omega_S}(s_T) = s_T$.

It can be shown [4] that in the case under consideration the operator weights $\boldsymbol{\pi}_v(t,S)$ and $\boldsymbol{\pi}_v(t,B)$ for the underlying security and the pure discount bond determined by equation (14) and relationship (15) are given by

$$\boldsymbol{\pi}_v(t,S) = \boldsymbol{v}_{\boldsymbol{s}}(t) \quad \text{and} \quad \boldsymbol{\pi}_v(t,B) = \boldsymbol{v}(t) - s\,\boldsymbol{v}_{\boldsymbol{s}}(t), \quad t \in \mathcal{T}, \tag{16}$$

where the subscript denotes the partial derivative.

In this regard, the portfolio operators $\hat{\pi}(t,S)$ and $\hat{\pi}(t,B)$ on Π are given by

$$\hat{\pi}(t,S) = \frac{\partial}{\partial s} \quad \text{and} \quad \hat{\pi}(t,B) = \boldsymbol{I} - s\,\frac{\partial}{\partial s},$$

where $\boldsymbol{I}$ is the identity operator on Π.

It is clear that the portfolio operators $\hat{\pi}(t,B)$ and $\hat{\pi}(t,S)$ on Π and linear operators on Π of multiplication by arbitrary admissible functions in Π do not commute in general:

$$[\hat{\pi}(t,B), h(\hat{s})] = -\hat{s}\,h_s(\hat{s}) \quad \text{and} \quad [\hat{\pi}(t,S), h(\hat{s})] = h_s(\hat{s}),$$

where $h(\hat{s})$ is the operator on Π of multiplication by the function $h = h(s)$ in Π, and the subscript denotes the partial derivative.

We note that the linear operators $\hat{\pi}(t,B)$, $\hat{\pi}(t,S)$, $\hat{s}$ and $\boldsymbol{I}$ on Π generate a Lie algebra similar to that in quantum mechanics with the nonzero structure constants determined by

$$[\hat{\pi}(t,B),\hat{\pi}(t,S)] = \hat{\pi}(t,S), \quad [\hat{\pi}(t,B),\hat{s}] = -\hat{s}, \quad \text{and} \quad [\hat{\pi}(t,S),\hat{s}] = \boldsymbol{I},$$

where $\hat{s}$ is the linear operator on Π of multiplication by the argument s.

It can be also shown [4] that in the case under consideration the evolution equation in (13) with the operator weights given by (16) is nothing but the Black-Scholes equation in (3). Moreover, the dynamically replicating portfolio can be chosen as the underlying security itself and the pure discount bond according to the standard delta hedging. We comment that in the derivation of the Black-Scholes equation we expressed the price of the underlying security in terms of the dividend yield $d(s,t)$ and used noncommutative analysis.

To the best of our knowledge the presented derivation of the Black-Scholes equation in seed form was encountered by Garman in his pioneering work [2].

14 The generators of a market environment as quasidifferential operators

The concept of a quasidifferential operator was introduced by the author in [13]. Roughly speaking, a quasidifferential operator is a linear operator that can be approximated in an appropriate sense by a sequence of linear differential operators, called a defining sequence, of increasing orders.

It was shown in [4] that each generator $\boldsymbol{L}(t)$ of a market environment $\mathbf{V}$ can be formally represented as the following quasidifferential operator

$$\boldsymbol{L}\left(t,\hat{s},\frac{\partial}{\partial s}\right) = \sum_{m=0}^{\infty} \boldsymbol{L}_m(t,\hat{s})\,\frac{\partial^m}{\partial s^m},$$

with the defining sequence given by

$$\boldsymbol{L}_M(t) = \sum_{m=0}^{M} \boldsymbol{L}_m(t,\hat{s})\,\frac{\partial^m}{\partial s^m},$$

where the operator coefficients $\boldsymbol{L}_m(t,\hat{s})$ are defined by

$$\boldsymbol{L}_m(t,\hat{s}) = \frac{1}{m!}\left((-\mathbf{ad}_{\hat{s}})^m\,\boldsymbol{L}(t)\right)\mathbf{1},$$

with $\mathbf{ad}_A B$ standing for the commutator $[A,B]$ of linear operators A and B.

We call a vector bundle $\mathfrak{A}$ with the fiber $\mathbb{R}$ and the base space $\mathbb{P} = \mathbb{R}_{++} \times \mathcal{T}$ an *account bundle*. We call the base space $\mathbb{P}$ a *price-time*.

Define the action of the gauge group $\mathfrak{G}$ on the fiber $\mathbb{R}$ over (s,τ) in $\mathbb{P}$ as multiplication by $\mathfrak{g}(\tau,s_\tau) > 0$ so that the value $v(\tau,s_\tau)$ in the fiber $\mathbb{R}$ over (s,τ) in $\mathbb{P}$ of a European contingent claim with inception time t and expiration time T with $t \le \tau \le T$ is transformed into $\mathfrak{g}^{-1}(\tau,s_\tau)\,v(\tau,s_\tau)$. In turn, according to the evolution equation in (2), $\mathfrak{g}^{-1}(\tau,s_\tau)\,v(\tau,s_\tau)$ is also the value of a European contingent claim but in a different market environment whose generators are determined by the action of the gauge group $\mathfrak{G}$ as follows

$$\boldsymbol{L}(t,\hat{s},\frac{\partial}{\partial s}) \xrightarrow{\mathfrak{g}_{\boldsymbol{L}}} \boldsymbol{L}\big(t,\hat{s},\frac{\partial}{\partial s}+\frac{\mathfrak{g}_{\boldsymbol{s}}(t)}{\mathfrak{g}(t)}\big) + \frac{\mathfrak{g}_{\boldsymbol{t}}(t)}{\mathfrak{g}(t)} \quad \text{and}$$
$$v(t) \xrightarrow{\mathfrak{g}_{\boldsymbol{\mathcal{V}}}} \mathfrak{g}^{-1}(t)v(t).$$

Then the evolution equation in (2) can be represented in the following covariant form

$$\nabla_t\, v + \boldsymbol{L}(t,\hat{s},\nabla_s)\, v = 0,$$

where ∇_t and ∇_s are the covariant derivatives defined by

$$\nabla_t = \frac{\partial}{\partial t} + \frac{\mathfrak{g}_{\boldsymbol{t}}(t)}{\mathfrak{g}(t)} \quad \text{and} \quad \nabla_s = \frac{\partial}{\partial s} + \frac{\mathfrak{g}_{\boldsymbol{s}}(t)}{\mathfrak{g}(t)}\cdot$$

It is clear that the related connection on the account bundle $\mathfrak{A}$ is trivial and hence the price-time $\mathbb{P}$ is a flat space.

15 The generators of a belief as quasidifferential operators

It was shown in [4] that each generator $\boldsymbol{\mathfrak{L}}(t)$ in $\boldsymbol{\mathfrak{L}}$ from $\mathbb{L}$ in the market populace $\mathbb{H} = \mathbb{L} \times \mathbb{U}$ can be formally represented as the following quasidifferential operator

$$\boldsymbol{\mathfrak{L}}\left(t,\hat{s},\frac{\partial}{\partial s}\right) = \sum_{m=1}^{\infty} \boldsymbol{\mathfrak{L}}_m(t,\hat{s})\,\frac{\partial^m}{\partial s^m},$$

with the defining sequence given by

$$\boldsymbol{\mathfrak{L}}_M(t) = \sum_{m=1}^{M} \boldsymbol{\mathfrak{L}}_m(t,\hat{s})\,\frac{\partial^m}{\partial s^m},$$

where the operator coefficients $\boldsymbol{\mathfrak{L}}_m(t,\hat{s})$ are defined by

$$\boldsymbol{\mathfrak{L}}_m(t,\hat{s}) = \frac{1}{m!}\left((-\mathbf{ad}_{\hat{s}})^m\,\boldsymbol{\mathfrak{L}}(t)\right)\mathbf{1}.$$

The transformations $\hat{\boldsymbol{g}} = (\hat{\boldsymbol{g}}_{\boldsymbol{\mathfrak{L}}},\hat{\boldsymbol{g}}_{\boldsymbol{\mathcal{U}}})$ in the beliefs-preferences gauge symmetry group $\hat{\boldsymbol{\mathcal{G}}}$ take the following form

$$\boldsymbol{\mathfrak{L}}\left(t,\hat{s},\frac{\partial}{\partial s}\right) \xrightarrow{\hat{\boldsymbol{g}}_{\boldsymbol{\mathfrak{L}}}} \boldsymbol{\mathfrak{L}}\left(t,\hat{s},\frac{\partial}{\partial s}+\frac{\mathfrak{g}_{\boldsymbol{s}}(t)}{\mathfrak{g}(t)}\right) + \frac{\mathfrak{g}_{\boldsymbol{t}}(t)}{\mathfrak{g}(t)} \quad \text{and}$$
$$\boldsymbol{u'}(t) \xrightarrow{\hat{\boldsymbol{g}}_{\boldsymbol{\mathcal{U}}}} \mathfrak{g}^{-1}(t)\boldsymbol{u'}(t).$$

The evolution equation in (5) can be represented in the following covariant form

$$\nabla_t\, v + \boldsymbol{\mathfrak{L}}(t,\hat{s},\nabla_s)\, v = 0,$$

where ∇_t and ∇_s are the covariant derivatives defined by

$$\nabla_t = \frac{\partial}{\partial t} + \frac{\mathfrak{g}_{\boldsymbol{t}}(t)}{\mathfrak{g}(t)} \quad \text{and} \quad \nabla_s = \frac{\partial}{\partial s} + \frac{\mathfrak{g}_{\boldsymbol{s}}(t)}{\mathfrak{g}(t)}\cdot$$

Finally, we note that relationship (7) can be represented as follows

$$\boldsymbol{L}\left(t,\hat{s},\frac{\partial}{\partial s}\right)=\mathfrak{L}\left(t,\hat{s},\frac{\partial}{\partial s}+\frac{\boldsymbol{u}'_{\boldsymbol{s}}(t)}{\boldsymbol{u}'(t)}\right)+\frac{\boldsymbol{u}'_{\boldsymbol{t}}(t)}{\boldsymbol{u}'(t)}.$$

It is easy to see with the help of the preceding relationship that the beliefs-preferences gauge symmetry group $\hat{\boldsymbol{\mathcal{G}}}$ can be viewed as a subgroup of the gauge group $\mathfrak{G}$ introduced in the previous section. This justifies the terminology introduced earlier in the article for $\hat{\boldsymbol{\mathcal{G}}}$ as the beliefs-preferences gauge symmetry group.

16 The method of quasidifferential operators

We present the method of quasidifferential operators proposed by the author in [4] for the approximate valuation and dynamic replication of European contingent claims in the case when the generators of the beliefs of market participants are quasidifferential operators.

Suppose that the set of European contingent claims $\{\mathfrak{v}_{S_i} : i=0,\dots,N\}$ with inception time t, expiration time T and the payoffs $v_T^{S_i}$ such that $v_T^{S_i}(s_T)=s_T^i$ is traded for each t and T in $\mathcal{T}$ with $t\le T$.

Consider the portfolios $\{\mathfrak{p}_i : i=0,\dots,N\}$ such that each $\mathfrak{p}_i$ is itself a portfolio of the European contingent claims $\mathfrak{v}_{S_j}$, $j=0,\dots,i$ with operator weights $\binom{i}{j}(-1)^{i-j}\,s_t^{i-j}$. We comment that the values $p_T^{(i)}=p_T^{(i)}(s_T)$ in Π of the portfolios $\mathfrak{p}_i$, $i=0,\dots,N$, at expiration time T are given by

$$p_T^{(i)}=\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\,s_t^{i-j}\,s_T^j=(s_T-s_t)^i,\quad i=0,\dots,N.$$

We note that the portfolios, or equivalently European contingent claims $\mathfrak{p}_i$, $i=0,\dots,N$, are effectively traded and that $\mathfrak{p}_i$ with $i=0$ is nothing but the pure discount bond.

Following [4] we will show that the set of the portfolios $\{\mathfrak{p}_i : i=0,\dots,N\}$ approximately, in the sense to be specified below, dynamically replicates each European contingent claim $\mathfrak{v}$ with inception time t, expiration time T and payoff v_T.

Denote by $\boldsymbol{\pi}_v(t,P_i)$ the operator weights of the portfolios $\mathfrak{p}_i$, $i=0,\dots,N$, in the dynamically replicating portfolio for the European contingent claim $\mathfrak{v}$ with the value $v=v(t)$.

Approximating each generator $\mathfrak{L}(t)$, viewed formally as a quasidifferential operator, by differential operators of increasing orders from its defining sequence

$$\mathfrak{L}_M(t)=\sum_{m=1}^{M}\mathfrak{L}_m(t,s)\,\frac{\partial^m}{\partial s^m},$$

we arrive [4] with the help of equation (14) and relationship (15) at the following expressions for the operator weights $\boldsymbol{\pi}_v(t,P_i)$:

$$\boldsymbol{\pi}_v(t,P_i)=\frac{1}{i!}\,\boldsymbol{v}^{(i)}(t),\quad i=0,\dots,N, \tag{17}$$

where N is equal to $M-1$, and $\boldsymbol{v}^{(i)}(t)$ is the linear operator on Π of multiplication by the function $\frac{\partial^i}{\partial s^i}v(t)$ in Π.

The corresponding portfolio operators $\hat{\pi}(t,P_i)$ on Π are given by

$$\hat{\pi}(t,P_i)=\frac{1}{i!}\,\frac{\partial^i}{\partial s^i}.$$

It is clear that the portfolio operators $\hat{\pi}(t,P_i)$ on Π and linear operators on Π of multiplication by arbitrary admissible functions in Π do not commute in general:

$$[\hat{\pi}(t,P_i),h(\hat{s})]=\frac{1}{i!}\sum_{j=0}^{i-1}j!\binom{i}{j}h^{(i-j)}(\hat{s})\,\hat{\pi}(t,P_j),$$

where $h(\hat{s})$ is the operator on Π of multiplication by the function $h=h(s)$ in Π, and $h^{(i)}$ stands for $\frac{\partial^i}{\partial s^i}h$.

It can be shown [4] that the Cauchy problem in (13) with the operator weights $\boldsymbol{\pi}_v(t,P_i)$ given by (17) takes the following form

$$\begin{aligned}&\frac{\partial}{\partial t}v_M+\mathfrak{L}_M(t,s)\frac{\partial^M}{\partial s^M}\,v_M\\&\quad-\sum_{i=1}^{M-1}\left(\frac{s^i}{i!}\left(\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\,\mathfrak{d}_j(s,t)\right)\right)\frac{\partial^i}{\partial s^i}v_M\\&\quad-r(s,t)v_M=0,\quad t<T,\\&\qquad v(T)=v_T,\end{aligned}$$

where $v_M=v_M(t)$ is the approximate value of the European contingent claim $\mathfrak{v}$ with inception time t, expiration time T and payoff v_T, and $\mathfrak{d}_i(s,t)$ are given by

$$\mathfrak{d}_i(s_t,t)\,s_t^i=\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t,T)|_{t=T}\,s_T^i.$$

It is clear that $\mathfrak{d}_i(s,t)$ with $i=0$ and $i=1$ are the interest rate $r(s,t)$ and the dividend yield $d(s,t)$ in terms of the underlying security being a stock.

If in the preceding Cauchy problem M goes to infinity then under the assumption that the term

$$\mathfrak{L}_M(t,s)\frac{\partial^M}{\partial s^M}\,v_M(t)$$

tends in an appropriate sense to the zero function we obtain the following Cauchy problem for the value $v=v(t)$ of the European contingent claim $\mathfrak{v}$ with inception time t, expiration time T and payoff v_T:

$$\begin{aligned}&\frac{\partial}{\partial t}v-\sum_{i=1}^{\infty}\left(\frac{s^i}{i!}\left(\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\,\mathfrak{d}_j(s,t)\right)\right)\frac{\partial^i}{\partial s^i}v\\&\quad-r(s,t)v=0,\quad t<T,\\&\qquad v(T)=v_T,\end{aligned}$$

In this case each generator $\boldsymbol{L}(t)$ of a market environment $\mathbf{V}$ can be formally represented as the following quasidifferential operator

$$\boldsymbol{L}(t) = -\sum_{i=0}^{\infty}\left(\frac{s^i}{i!}\left(\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\,\mathfrak{d}_j(s,t)\right)\right)\frac{\partial^i}{\partial s^i}.$$

Finally, we note that the method of quasidifferential operators for the approximate valuation and dynamic replication of European contingent claims can be viewed as a generalization of the method based on the Edgeworth expansion (see [15]).

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References

1. D. Duffie, *Security Markets: Stochastic Models* (Academic Press, San Diego, California, 1988).
2. M.B. Garman, J. Finance **40**, 847 (1985).
3. R.C. Merton, *Continuous Time Finance* (Blackwell, Cambridge, Massachusetts, 1990).
4. V.A. Kholodnyi, *Beliefs-Preferences Gauge Symmetry Group and Replication of Contingent Claims in a General Market Environment* (IES Press, Research Triangle Park, North Carolina, 1998). (Based on: V.A. Kholodnyi, *Beliefs-Preferences Gauge Symmetry Group and Replication of Contingent Claims in a General Market Environment*, IES preprint, 1995.)
5. V.A. Kholodnyi, J.F. Price, *Foreign Exchange Option Symmetry* (World Scientific, River Edge, New Jersey, 1998).
6. V.A. Kholodnyi, J.F. Price, *Foundations of Foreign Exchange Option Symmetry* (IES Press, Fairfield, Iowa, 1998).
7. V.A. Kholodnyi, J.F. Price, *Foreign Exchange Option Symmetry Based on Domestic-Foreign Payoff Invariance, Proceedings of the Conference on Computational Intelligence for Financial Engineering*, edited by J.F. Marshal, R.J. Marks (IEEE/IAFE, New York, 1997), pp. 164–170.
8. V.A. Kholodnyi, J.F. Price, Nonlinear Analysis **47**, 5885 (2001).
9. V.A. Kholodnyi, Nonlinear Analysis **30**, 5059 (1997).
10. V.A. Kholodnyi, *A Semilinear Evolution Equation for General Derivative Contracts*, In John F. Price, editor, *Derivatives and Financial Mathematics* (Nova Science Publishers, Inc., Commack, New York, 1997), pp. 119–138.
11. V.A. Kholodnyi, *On the Linearity of European, Bermudan and American Options with General Time-Dependent Payoffs in Partial Semimodules* (IES preprint, 1995).
12. V.A. Kholodnyi, *Universal Contingent Claims* (IES preprint, 1995).
13. V.A. Kholodnyi, *Invention and Elaboration of the Method of Quasidifferential Operators for the Analysis of Slow-Wave Systems*, Ph.D. Thesis, Moscow Institute of Electronics and Mathematics, 1990.
14. V.A. Kholodnyi, *Beliefs-Preferences Gauge Symmetry and Noncommutativity as the Formalization of Randomness of the Prices of Underlying Securities* (IES preprint, 1998).
15. V.A. Kholodnyi, *Beliefs-Preferences Gauge Symmetry and the Method of Quasidifferential Operators for the Approximate Valuation and Dynamic Replication of Contingent Claims as an Extension of the Edgeworth Expansion* (IES preprint, 1998).
16. B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov, *Modern Geometry – Methods and Applications* (Springer-Verlag, New York, 1992).
17. J.D. Dollard, C.N. Friedman, *Product Integration* (Addison-Wesley, Reading, Massachusetts, 1979).
18. V.P. Maslov, *Operational Methods* (Mir, Moscow, Russia, 1976).